Convergence of Series I: In-Class Exercises

At this point, we have the following resources for determining the convergence/divergence of series:
The n^{th} term test
Evaluating the sequence of partial sums
The Integral Test
Geometric Series
p-Series: $\sum_{n=0}^{\infty} \frac{1}{n^p}$
Comparison Test
Limit Comparison Test

$$\sum_{n=0}^{\infty} \frac{1}{e^{2n}}$$

$= \sum_{n=0}^{\infty} \frac{1}{(e^2)^n} = \sum_{n=1}^{\infty} \frac{1}{e^{2n-1}} = \frac{1}{1-1/e^2}$ The series converges because it is a geometric series with $r = \frac{1}{e^2}$ and $|\frac{1}{e^2}| < 1$

$$\sum_{n=0}^{\infty} \frac{n^3}{n(n+2)}$$

Note that $\lim_{n\to\infty} \frac{n^3}{n(n+2)} = \infty$ since the degree of the numerator is greater than the degree of the denominator. We conclude that the series diverges by the n^{th} term test.

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{6^n\sqrt{n}}$$

We notice that the dominant terms in the numerator and denominator are 2^n and 3^n, respectively, so we expect this series to behave a lot like the geometric series $\sum_{n=1}^{\infty} (\frac{1}{3})^{n-1}$ in the long term. In fact, for $n \geq 1$, $\frac{2^{n-1}}{6^n\sqrt{n}} = \frac{1}{6}(\frac{1}{3})^{n-1}\frac{1}{\sqrt{n}} < (\frac{1}{3})^{n-1}$. We know that $\sum_{n=1}^{\infty} (\frac{1}{3})^{n-1}$ is a convergent geometric series since $|\frac{1}{3}| < 1$, so $\sum_{n=1}^{\infty} \frac{2^{n-1}}{6^n\sqrt{n}}$ converges by the Comparison Test.

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$$

We suspect that this series will diverge because ln approaches 0 more slowly than $\frac{1}{n}$ and the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. To show that this is indeed the case, we use limit comparison:

$\displaystyle\lim_{n\to\infty} \frac{\frac{1}{\ln(n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \infty$. This application of the limit comparison test shows that $\frac{1}{n}$ indeed approaches 0 more quickly than $\frac{1}{\ln(n)}$, and therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ diverges.

$$\sum_{n=1}^{\infty} \frac{3}{4^n+5}$$

We suspect that the dominant term in this series will be the $\frac{1}{4^n}$. Therefore, a comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^n}$ comes to mind as a possibility.

Indeed: $\frac{3}{4^n+5} \leq \frac{3}{4^n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^n} = \sum_{n=1}^{\infty} (\frac{3}{4}) \frac{1}{4^{n-1}}$ is a geometric series with $|r| = |\frac{1}{4}| < 1$ Since $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^n}$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^n+5}$ converges by the Comparison Test.

Note: Several of the above series can also be tested with the Integral Test. If you did not use this method, try it for a few.

Solution to the quiz question:

$$\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$$

$s_1 = 1 - \frac{1}{2}$

$s_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$ $s_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$

This pattern of cancellation continues, yielding $\displaystyle s_k = \sum_{n=1}^{k} (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{k+1}$

By definition, $\displaystyle\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1}) = \lim_{k\to\infty} s_k = \lim_{k\to\infty} (1 - \frac{1}{k+1}) = 1$

Therefore, the series converges to 1.